Journal of Engineering Physics and Thermophysics, Vol. 73, No. 2, 2000

STOCHASTIC MODELING OF THE DEVELOPMENT OF THE TAYLOR INSTABILITY IN LIQUID FILTRATION IN A POROUS MEDIUM

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UDC 532.546

A modified model of stochastic growth that is controlled by the pressure gradient is used for a numerical study of the instability of the displacement front of liquids with different densities and viscosities in a porous medium. Consideration is given to the effect of permeability and porosity nonuniformities of the medium on the displacement front.

Introduction. The Taylor instability of the displacement front of liquids in a porous medium develops when a heavy liquid displaces a lightweight liquid residing under the former [1] or a more viscous liquid is displaced by a less viscous one [2, 3]. A study of the hydrodynamic Taylor instability is of both scientific and practical importance that is related to the appearance of flow instabilities in waterflooding of petroliferous strata, propagation of liquid wastes in subsurface aquiferous strata, and some chemical-technology processes.

The application of analytical methods to the study of Taylor instability encounters serious difficulties and is restricted to studying the initiation and asymptotic stage of the development of instability for fairly simple displacement models. The use of numerical methods permits a description of all stages of the development of instability and an examination of more general filtration models [4, 5]. However, deterministic numerical models of unstable displacement have a major drawback that lies in the need to impose artificially the initial flow disturbance in the form of an uneven front or a nonuniform distribution of the medium permeability. Here, the formation of the front instability depends on the type of selected disturbance. Under actual conditions, the flow instability arises spontaneously as a consequence of the presence of microheterogeneites of the porous medium and fluctuations of the liquid motion in the pores. Spontaneous and permanent initiation of the instability can be realized by introducing a stochastic factor into the numerical model. L. Paterson [6] and L. P. Kadanoff [7] proposed the use of randomly moving particles for modeling unstable liquid displacement in a porous medium. The application of the random-walk method is based on the similarity of the equations that describe the probability density distribution for the occurrence of a randomly moving particle in space and the liquid pressure in the medium. The region occupied by the displacing liquid corresponds to an aggregate of particles that grows from an injection circuit. The advance of the displacement front is simulated by attachment, to an existing aggregate, of a new randomly moving particle released from a surface that corresponds to a discharge circuit. A further elaboration of the stochastic approach to the description of an unstable displacement was the development of a model of growth controlled by the pressure gradient [8, 9]. In this model, the probability of advance of the displacement front depends on the normal component of the pressure gradient.

The aforementioned stochastic models describe the space-time dynamics of the displacement of liquids with various viscosity ratios in a medium that is uniform in its filtration parameters. However, they disregard gravitational effects arising in the filtration of liquids with different densities and cannot be used for modeling the gravitational Taylor instability.

The current work deals with a generalization of the model of growth controlled by the pressure gradient for describing the motion of the displacement front of liquids with different densities and viscosities in a

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Seversk Technological Institute of Tomsk Polytechnic University, Tomsk, Russia. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 73, No. 2, pp. 267-273, March–April, 2000. Original article submitted March 11, 1999.

1062-0125/00/7302-0259$25.00

porous medium with a nonuniform permeability and porosity distribution. The proposed model is used for studying the Taylor instability for different relationships between the viscous and gravitational forces.

Formulation of the Stochastic Displacement Model. Consideration is given to the displacement, from a porous medium, of a liquid with viscosity μ_1 and density ρ_1 by another liquid with viscosity μ_2 and density ρ_2. Both liquids are assumed to be incompressible. Their motion is described in the approximation of piston displacement. The porous medium is divided into two regions containing the displaced and displacing liquids. Capillary forces at the liquid-liquid interface are disregarded. The filtration rate $\vec{v}$ for each liquid obeys Darcy's law for viscous flow in a porous medium

$$\vec{v_l} = -\frac{k}{\mu_l}(\mathrm{grad}\, P_l - \vec{g}\rho_l)\,, \quad l = 1, 2\,. \tag{1}$$

The pressure distribution in each region satisfies the equation obtained from the condition of continuity of incompressible liquid flow ($\mathrm{div}\, \vec{v} = 0$) and Darcy's law (1)

$$\mathrm{div}\left(\frac{k}{\mu_l}(\mathrm{grad}\, P_l - \vec{g}\rho_l)\right) = 0\,, \quad l = 1, 2\,. \tag{2}$$

At the liquid-liquid interface Γ, the pressures and the filtration velocity components normal to the interface are equal:

$$P_1|_\Gamma = P_2|_\Gamma\,, \tag{3}$$

$$(\vec{v_1}\cdot\vec{n})|_\Gamma = (\vec{v_2}\cdot\vec{n})|_\Gamma\,. \tag{4}$$

Conditions (3) and (4) in conjunction with the boundary conditions on the discharge and injection circuits permit calculation of the pressure distribution in the considered region using Eq. (2).

The velocity of the displacement front is determined by the normal component U_n of the actual flow velocity

$$U_n = \frac{(\vec{v_1}\cdot\vec{n})}{m} = \frac{(\vec{v_2}\cdot\vec{n})}{m}\,. \tag{5}$$

From the condition of flow continuity (4) it follows that the velocity of the displacement front can be calculated using the pressure distribution in both the displacing and displaced liquids. The subsequent discussion draws on the latter. Within the framework of the stochastic approach, Eq. (5) acquires a probabilistic meaning. The probability density ω for advance of the displacement front at one or another place is taken to be proportional to the normal component of the actual flow velocity U_n if $U_n > 0$ and equal to zero for $U_n \le 0$:

$$\omega = \theta(U_n)\, U_n Z^{-1}\,. \tag{6}$$

The normalization factor Z is defined by the integral over the liquid-liquid interface

$$Z = \int_\Gamma \theta(U_n)\, U_n\, dS\,. \tag{7}$$

The mean velocity of various front regions that corresponds to the probability distribution (6) coincides with the actual velocity (5). The introduction of the condition $U_n > 0$ into Eqs. (6) and (7) renders the displacement irreversible. The replacement of the deterministic and continuous description of the motion of the displacement front by a probabilistic and discrete one is a key element of the stochastic approach.

The stochastic model is employed for describing two-dimensional displacement of liquids in a rectangular region ($0 \le x \le L_x$, $0 \le y \le L_y$). The direction of the vector of acceleration due to gravity $\vec{g}$ coincides with the direction of the Y coordinate axis. Two types of boundary conditions are examined. In the first case, the flow direction is perpendicular to the vector of the acceleration due to gravity. The pressure distribution at the outlet ($x = L_x$) and the inlet ($x = 0$) is specified by the relations

$$P_1 = P_{\text{out}} + g\rho_1 y\,, \quad x = L_x\,, \quad 0 \le y \le L_y\,,$$
$$P_2 = P_{\text{in}} + g\rho_2 y\,, \quad x = 0\,, \quad 0 \le y \le L_y\,, \tag{8}$$

and the lateral boundaries ($y = 0$, $y = L_y$) are assumed to be impermeable:

$$\frac{\partial P}{\partial y} = 0\,, \quad y = 0\,, \quad y = L_y\,, \quad 0 \le x \le L_x\,. \tag{9}$$

In the second case, the flow direction coincides with the acceleration due to gravity: at the inlet ($y = 0$) and the outlet ($y = L_y$), constant pressures P_{in} and P_{out}, respectively, are prescribed and the boundaries $x = 0$, $x = L_x$, $0 \le y \le L_y$ are impermeable.

Numerical Realization of the Model. Model discretization is carried out with the aid of a uniform rectangular grid with a period h. For each node of the grid (i, j), there is the corresponding element of the medium with the permeability $k_{i,j}$ and the porosity $m_{i,j}$. The node state of the node is determined by the pressure $P_{i,j}$, viscosity $\mu_{i,j}$, and density $\rho_{i,j}$ of the liquid that occupies the pore space. The elements of the medium corresponding to the grid nodes are regarded as completely filled with the displacing or displaced liquid, i.e., the displacement front passes between neighboring nodes. The front advances in discrete steps. The probability $W_{i,j,i',j'}$ of front movement from the node (i, j) to the neigboring node (i', j') with the latter being filled with the displacing liquid is defined by the discrete analog of Eq. (6)

$$W_{i,j,i',j'} = \theta\,(U_{i,j,i',j'})\, U_{i,j,i',j'}\, Z^{-1}\,.$$

The velocity of front advance between the nodes is defined by the expression

$$U_{i,j,i',j'} = \frac{T_{i,j,i',j'}}{m_{i',j'}}\,(P_{i,j} - P_{i',j'} + G_{i,j,i',j'})\,, \quad G_{i,j,i',j'} = \frac{gh}{2}\,(\rho_{i,j} + \rho_{i',j'})\,(j' - j)\,.$$

The hydraulic conductivity $T_{i,j,i',j'}$ between the nodes is calculated by the formula

$$T_{i,j,i',j'} = \frac{2k_{i,j}\, k_{i',j'}}{k_{i,j}\,\mu_{i,j} + k_{i',j'}\,\mu_{i',j'}}\,.$$

The normalization factor Z is defined by the expression

$$Z = \sum_{i,j,i',j'} \theta\,(U_{i,j,i',j'})\, U_{i,j,i',j'}\,,$$

where the summation is carried out over all possible directions and regions of the displacement front. The pressure distribution is calculated based on a finite-difference approximation of Eq. (2) with account for the conditions of continuity at the liquid-liquid interface

$$\sum_{i',j'} T_{i,j,i',j'}\,(P_{i,j} - P_{i',j'} + G_{i,j,i',j'}) = 0\,. \tag{10}$$

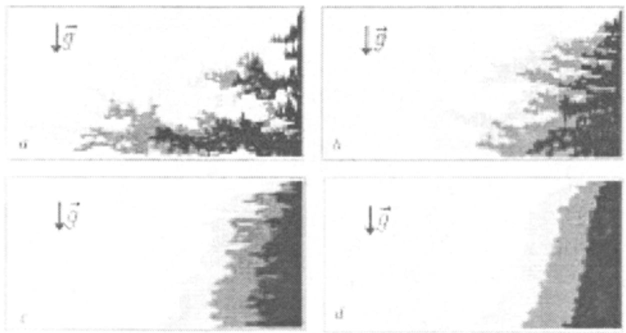

Fig. 1. Typical modeling results for displacement at different ratios of the liquid viscosities (R = 0.8) (the gray shades show successive positions of the front): a) M = 100; b) 10; c) 1; d) 0.1.

Finite-difference equation (10) is solved by an iteration method using a five-point "cross" pattern [10]. The pressure distribution is recalculated after each advance of the front by one node of the grid.

The time interval Δt corresponding to one calculational step is determined by the volume of the pore space $h^2 m_{i',j'}$ occupied by the displacing liquid in the given step and the liquid flow Q through the injection circuit:

$$\Delta t = \frac{h^2 m_{i',j'}}{Q} .$$

The liquid flow Q is calculated by summing up the flows between the nodes (i, j) that belong to the injection circuit and the neighboring nodes (i', j'):

$$Q = \sum_{i,j,i',j'} T_{i,j,i',j'} (P_{i,j} - P_{i',j'} + G_{i,j,i',j'}) .$$

Results and Discussion. The numerical realization of the model was used for studying the effect of the liquid viscosities and densities on the development of the instability of the displacement front. Consideration was also given to effects related to the nonuniformity of the permeability and porosity distributions of the medium. Calculations were performed on a grid that measured 120 × 60 nodes.

Analysis of the results of the numerical modeling reveals that the advance of the displacement front in a homogeneous medium is determined by the ratios of the liquid viscosities $M = \mu_1/\mu_2$ and densities $R = \rho_1/\rho_2$, the filtration velocity, and its direction relative to the vector of the acceleration due to gravity. In the case where the filtration velocity is perpendicular to the acceleration due to gravity (boundary conditions (8) and (9)) the formation of the front instability is determined primarily by the ratio of the liquid viscosities M. Figure 1 presents typical results of the modeling of the advance of the displacement front in a homogeneous medium for various viscosity ratios M and the same density ratio R = 0.8. If the viscosity of the displacing liquid is lower than the viscosity of the displaced liquid ($M > 1$), the stochastic variations of the front shape are increased, and instability of the liquid flow develops, which manifests itself in the formation of viscous fingers (Fig. 1a). The pressure redistribution in the medium causes dominating viscous fingers to suppress the growth of those lagging behind. As the fingers develop, they expand and split into smaller ones. With increase in the viscosity of the displacing liquid, the viscous fingers thicken and their development slows down (Fig. 1b). Similar effects were observed in experimental investigations of the displacement instability in a porous medium and in Hele–Shaw cells [11, 12]. If the values of the viscosities are close ($M \approx 1$), stochastic disturbances of the flow do not develop and are not suppressed. However, random superposition of the disturbances leads to formation of a nonuniform front (Fig. 1c). This was observed in experiments on the mutual displacement of homogeneous liquids and is attributed to the nonuniformity of the velocity field as a consequence of the mi-

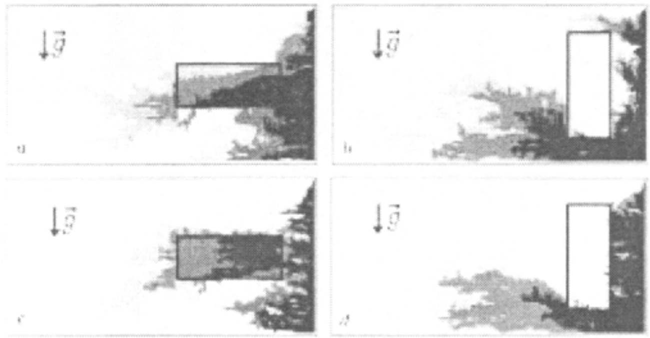

Fig. 2. Modeling results for unstable displacement in a heterogeneous medium (M = 10, R = 0.8): a) inclusion with an increased permeability (k_{inc}/k = 10); b) same with a decreased permeability (k_{inc}/k = 0.1), c) same with a decreased porosity (m_{inc}/m = 0.1); d) same with an increased porosity (m_{inc}/m = 10). The rectangles denote the inclusion boundaries.

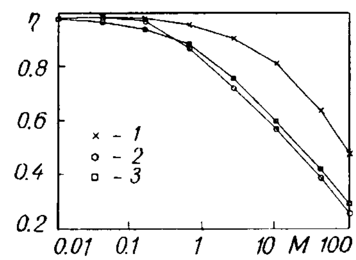

Fig. 3. Fraction η of the pore space filled with the displacing liquid at the instant of its reaching the discharge circuit vs. ratio M of the liquid viscosities (R = 1): 1) homogeneous medium; 2) in the presence of an inclusion with an increased permeability (k_{inc}/k = 10); 3) same with a decreased permeability (k_{inc}/k = 0.1).

croheterogeneity of the medium [13]. If the viscosity of the displacing liquid is higher than the viscosity of the displaced liquid ($M < 1$), the development of stochastically appearing protrusions and depressions of the front is suppresed by the pressure redistribution, and the line of the front is stabilized (Fig. 1d).

A difference in the densities of the displacing and displaced liquids affects the shape of the displacement front but does not change the condition of the instability development: $M > 1$. A decrease in the density ratio ($R < 1$) causes the heavier displacing liquid to flow under the displaced liquid. In the case of unstable displacement ($M > 1$), the development of viscous fingers at the bottom of the front speeds up. With the opposite density ratio ($R > 1$), the displacement picture is similar to within a turnover around the horizontal axis.

Nonuniformities of the permeability and porosity of the medium change the velocity field and affect the development of the instability of the liquid flow. An increase in the permeability, just like a decrease in the porosity, increases the actual velocity of the liquid flow. Therefore, inclusions with an increased permeability and a decreased porosity determine the predominant direction of development of viscous fingers. Conversely, inclusions with a decreased permeability and an increased porosity decrease the flow velocity and impede the formation of viscous fingers. Figure 2a, b presents typical results of modeling the formation of viscous fingers in the presence of rectangular inclusions with a permeability k_{inc} that is increased or decreased relative to the permeability k of the basic medium. Figure 2c, d shows the effect of inclusions with a porosity m_{inc} that is different from the porosity m of the medium on the displacement front. Clearly, the pictures of instability development in the presence of permeability and porosity nonuniformities are similar. This is caused by the fact that the formation of viscous fingers leads to a pressure redistribution that is similar for the two cases. Thus, inclusions determine the most probable places of formation of viscous fingers and change their rate of growth and shape.

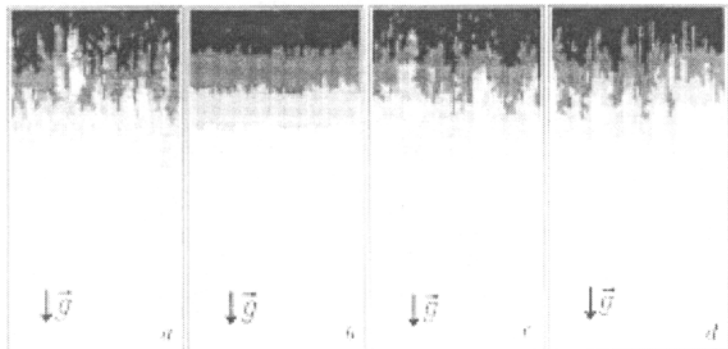

Fig. 4. Modeling results for the motion of the displacement front for various relationships between the viscous and gravitational forces: a) R = 0.2 and M = 1; b) 5 and 1; c) 0.2 and 0.5; d) 5 and 2.

An important characteristic of the efficiency of waterflooding of an oil field is the magnitude of the water-free extraction of oil, i.e., the volume fraction of oil displaced from the petroliferous stratum by the instant of appearance of water in the production well. In the given model, the fraction η of the pore space filled with the displacing liquid by the instant of its reaching the discharge circuit corresponds to the magnitude of the water-free extraction of oil. Figure 3 shows dependences of η on the ratio M of the liquid viscosities obtained in modeling the displacement in a homogeneous medium and in a medium with high-permeability (k_{inc}/k = 10) and low-permeability (k_{inc}/k = 0.1) inclusions (the position and size of the inclusions coincide with those shown in Fig. 2a). Conversion to an unstable mode of displacement ($M > 1$) entails a noticeable decrease in η. The presence of inclusions both with increased and with decreased permeability diminishes η in comparison with a homogeneous medium. However, with a stable mode of displacement ($M < 1$), the effect of the inhomogeneity of the permeability of the inclusions on η is markedly weaker.

In the case where the flow direction coincides with the vector of acceleration due to gravity, the character of the motion of the displacement front is governed by the balance of the viscous and gravitational forces. If the density of the displaced liquid is lower, and the viscosity is higher, than the corresponding parameters of the displacing liquid ($R < 1$, $M \geq 1$), the displacement front is always unstable. An example of the development of gravitational instability is shown in Fig. 4a. Similar pictures were observed in experiments on the two-dimensional displacement of a heavy liquid by a more lightweight one residing under the former [14]. With the opposite ratios of the densities and viscosities of the displaced and displacing liquids ($R \geq 1$, $M \leq 1$), the displacement front is stable (Fig. 4b). In the remaining cases, the motion of the front is determined by the relationship between the filtration velocity v and the critical velocity v_c that is obtained from a linear analysis of the stability [3]:

$$v_c = \frac{kg(\rho_1 - \rho_2)}{\mu_1 - \mu_2} .$$

If the instability is related to the gravitational forces ($R < 1$) and the viscous forces stabilize the front ($M < 1$), the instability of the displacement front develops at velocities below critical, $v < v_c$ (Fig. 5c). In the opposite case ($R > 1$, $M > 1$), the roles of the gravitational and viscous forces switch, and the instability of the front occurs at velocities above critical, $v > v_c$ (Fig. 4d).

Conclusion. The modified model of growth that is controlled by the pressure gradient adequately describes the main regularities of the development of Taylor instability in liquid filtration in a heterogeneous porous medium. Calculated results agree with experimental data on the unstable displacement of liquids in thin layers of porous media. The proposed model can be used for predicting the formation of waterflooding tongues when oil is displaced from heterogeneous strata and for assessing the dimensions of the region of propagation of liquid wastes in subsurface strata in solving geoenvironmental problems.

The work was carried out with support from the Russian Fund for Fundamental Research, grant No. 98-05-03150.

NOTATION

μ_l, viscosity; ρ_l, density; $\vec{v_l}$, filtration velocity, the subscript l = 1, 2 pertains to the displacing and displaced liquids, respectively; k, permeability of the medium; P, hydrostatic pressure; $\vec{q}$, acceleration due to gravity; U, actual flow velocity; m, porosity of the medium; $\vec{n}$, normal to the liquid-liquid interface directed from the displacing liquid to the displaced liquid; ω, probability density for displacement of the front; dS, element of the liquid-liquid interface; x, y, Cartesian coordinates; t, time; L_x, L_y, dimension of the modeling region along the X and Y axes; h, period of the rectangular grid; Z, normalization factor; Q, liquid flow through the injection circuit; M, ratio of the viscosities of the displaced and displacing liquids; R, ratio of densities of displaced and displacing liquids; v_c, critical velocity; θ, step function ($\theta(x)$ = 1 for $x > 0$ and $\theta(x)$ = 0 for $x \leq 0$); η, fraction of the pore space filled with the displacing liquid.

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